Solutions

MATH 1A Worksheet(April 4th)

1. Sketch the function graph of the following functions, including their (1) **interval of increase/decrease**, (2) **local extrema**, (3) **concavity, inflection points** and (4) **horizontal/vertical asymptotes**, if there is any.

   (a) $f(x) = x^3 - 12x + 2.$; $f'(x) = 3x^2-12$ inc: $(-\infty,-2)\cup(2,\infty)$ dec: $(-2,2)$ | $f''(x)=6x$ C.U. $[0,\infty)$ C.D. $(-\infty,0]$ infl. 0

   (b) $f(x) = x\sqrt{6-x}, x \leq 6.$

   (c) $f(x) = \ln(x^2+9).$

   (d) $f(x) = 2\sin\theta + \sin^2 2\theta, 0 \leq \theta \leq 2\pi.$

   (e) $f(x) = e^{-x^2/a}, a > 0.$

2. In this question, f is a differentiable function, with $f'(x_0) = 0$, i.e. x_0 is a critical number of f.

   (a) State the first derivative test with respect to the critical number x_0 of f, — if f' goes $- \to +$, local min; $+ \to -$ local max.

   (b) Suppose further, that f is second continuously differentiable, i.e. f' is differentiable and f'' is continuous. State the second derivative test with respect to the critical number x_0. $f''<0$, local max $f''>0$, local min.

   (c) If the first derivative works(that is, we can tell whether x_0 is a local maximum/local minimum/ neither from the first derivative test), will the second derivative always work? If so, explain why; if not, give some counter examples. $f(x)=x^4$ gives 0; $f(x)=x^{4/3}$

   (d) * If the second derivative test works, will the first derivative test always work? If so, explain why; if not, give some counter examples. yes, second d exists ⇒ first does. & parity ⇒ first inc or dec.

   (e) * In this part, we let $x_0 = 0$, and consider the following three functions,

$$f_1(x) = x^4 \sin\frac{1}{x}, f_2(x) = x^4(2+\sin\frac{1}{x}), f_3(x) = x^4(-2+\sin\frac{1}{x}), x \neq 0,$$

   and $f_1(0) = f_2(0) = f_3(0) = 0.$

   Show that f_1, f_2, f_3 are differentiable, and 0 is a critical number of these three functions. Check whether the first derivative test works for these functions. Tell whether 0 is (local maximum/local minimum/neither) for each of these three functions.

$f_1' = 4x^3\sin(1/x) - x^2\cos(\frac{1}{x})$; → First D fails b/c of oscillation

$f_1'' = 12x^2\sin(1/x) - 4x\cos(1/x) - 2x\cos(1/x) - \sin(1/x)$ doesn't exist at 0!

$f_2' = 4x^3(2+\sin(\frac{1}{x})) - x^2\cos(1/x)$; First D says local min

$f_2'' = 12x^2(2+\sin(1/x)) - 4x\cos(1/x) - 2x\cos(1/x) + \sin(1/x)$ No Change

$f_3' =$ ...

First D says local max.

B.) $f(x) = x\sqrt{6-x}$; $f'(x) = \sqrt{6-x} + \frac{x}{\sqrt{6-x}} = \frac{6-2x}{\sqrt{6-x}}$ Inc. $x < 3$

$x \le 6$ Dec. $x > 3$

$f''(x) = \frac{(-2)(\sqrt{6-x}) - (6-2x)\left(\frac{-1}{2\sqrt{6-x}}\right)}{6-x} = \frac{x(3-x) - 4(6-x)}{2(6-x)\sqrt{6-x}} = \frac{x(3-x)}{2(6-x)^{3/2}} - \frac{2}{\sqrt{6-x}}$

C.U.

C.D. all reals. <6

$-x^2 + 3x + 4x - 24$

$-x^2 + 7x - 24 < 0$

No Horiz. Asymptotes, No vertical.

c.) $\ln(x^2+9)$ No asympt.

$f'(x) = \frac{2x}{x^2+9}$ inc: $x>0$ dec: $x<0$

$f''(x) = \frac{2}{x^2+9} + \frac{2x(-1)(2x)}{(x^2+9)^2} = \frac{1}{(x^2+9)^2}(-4x^2 + 2x^2 + 18)$

$18 - 2x^2$ C.D. $\{x<3\} \cup \{x>3\}$

C.U. $x \in (-3, 3)$

d) $e^{-x^2/a}$ $a>0$ Inc: $(-\infty, 0)$ Dec: $(0, \infty)$

H.A. at $y=0$

C.U. $\{x < -\sqrt{a/2}\} \cup \{\sqrt{a/2} < x\}$

C.D. $(-\sqrt{a/2}, \sqrt{a/2})$

$f'(x) = e^{-x^2/a} \cdot \left(\frac{-2x}{a}\right)$

$f''(x) = \left(-\frac{2}{a}\right)e^{-x^2/a} + \frac{4x^2}{a^2}e^{-x^2/a}$

$= \frac{1}{a^2}(4x^2 - 2a)e^{-x^2/a}$